

GEOMETRIC DESCRIPTION OF THE PREDUALS OF ATOMIC COMMUTATIVE VON NEUMANN ALGEBRAS

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ABSTRACT. Strongly facially symmetric spaces isometrically isomorphic to the predual space of an atomic commutative von Neumann algebra are described.

1. INTRODUCTION

An important problem of the theory of operator algebras is a geometric characterization of state spaces of operator algebras. In the mid-1980s, Friedman and Russo wrote the paper [1] related to this problem, in which they introduced facially symmetric spaces, largely for the purpose of obtaining a geometric characterization of the predual spaces of JBW^* -triples admitting an algebraic structure. Many of the properties required in these characterizations are natural assumptions for state spaces of physical systems. Such spaces are regarded as a geometric model for states of quantum mechanics. In [2], it was proved that the preduals of von Neumann algebras and, more generally, JBW^* -triples are neutral strongly facially symmetric spaces.

The project of classifying facially symmetric spaces was initiated in [3], where a geometric characterization of complex Hilbert spaces and complex spin factors was given. The JBW^* -triples of ranks 1 and 2 and Cartan factors of types 1 and 4 were also described. Afterwards, Friedman and Russo obtained a description of atomic facially symmetric spaces [4]. Namely, they showed that a neutral strongly facially symmetric space is linearly isometric to the predual of one of the Cartan factors of types 1–6 provided that it satisfies four natural physical axioms, which hold for the predual spaces of JBW^* -triples. In the 2004 paper [5], Neal and Russo found geometric conditions under which a facially symmetric space is isometric to the predual of a JBW^* -triple. In particular, they proved that any neutral strongly facially symmetric space decomposes into a direct sum of atomic and nonatomic strongly facially symmetric spaces. This paper describes strongly facially symmetric spaces isometrically isomorphic to preduals of atomic commutative von Neumann algebras.

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2. PRELIMINARIES

Let Z be a real or complex normed space. We say that elements $x, y \in Z$ are *orthogonal* and write $x \diamond y$ if

$$\|x + y\| = \|x - y\| = \|x\| + \|y\|.$$

We say that subsets $S, T \subset Z$ are *orthogonal* and write $(S \diamond T)$ if $x \diamond y$ for all $(x, y) \in S \times T$. For a subset S of Z , we put

$$S^\diamond = \{x \in Z : x \diamond y, \forall y \in S\};$$

the set S^\diamond is called the *orthogonal complement* of S . A convex subset F of the unit ball

$$Z_1 = \{x \in Z : \|x\| \leq 1\}$$

is called a *face* if the relation

$$\lambda y + (1 - \lambda)z \in F, \text{ where } y, z \in Z_1, \lambda \in (0, 1),$$

implies $y, z \in F$. A face F of the unit ball is said to be *norm exposed* if

$$F = F_u = \{x \in Z : u(x) = 1\}$$

for some $u \in Z^*$ with $\|u\| = 1$. An element $u \in Z^*$ is called a *projective unit* if $\|u\| = 1$ and $u(y) = 0$ for all $y \in F_u^\diamond$ (see [1]).

Definition 2.1. ([1]). A norm exposed face F_u in Z_1 is called a *symmetric face* if there exists a linear isometry S_u from Z to Z such that $S_u^2 = I$ whose fixed point set coincides with the topological direct sum of the closure $\overline{sp}F_u$ of the linear hull of the face F_u and its orthogonal complement F_u^\diamond , i.e., with $(\overline{sp}F_u) \oplus F_u^\diamond$.

Definition 2.2. ([1]). A space Z is said to be *weakly facially symmetric (WFS)* if each norm exposed face in Z_1 is symmetric.

For each symmetric face F_u , contractive projections $P_k(F_u)$, $k = 0, 1, 2$, on Z are defined as follows. First, $P_1(F_u) = (I - S_u)/2$ is the projection onto the eigenspace corresponding to the eigenvalue -1 of the symmetry S_u . Next, $P_2(F_u)$ and $P_0(F_u)$ are defined as projections of Z onto $\overline{sp}F_u$ and F_u^\diamond , respectively; i.e., $P_2(F_u) + P_0(F_u) = (I + S_u)/2$. The projections $P_k(F_u)$ are called the *geometric Peirce projections*.

A projective unit u from Z^* is called a *geometric tripotent* if F_u is a symmetric face and $S_u^*u = u$ for the symmetry S_u corresponding to F_u . By \mathcal{GT} and \mathcal{SF} we denote the sets of all geometric tripotents and symmetric faces, respectively; the correspondence

$$\mathcal{GT} \ni u \mapsto F_u \in \mathcal{SF}$$

is one-to-one (see [6, Proposition 1.6]). For each geometric tripotent u from the dual *WFS* space Z , we denote the Peirce projections by

$$P_k(u) = P_k(F_u), k = 0, 1, 2.$$

We set

$$U = Z^*, Z_k(u) = Z_k(F_u) = P_k(u)Z, U_k(u) = U_k(F_u) = P_k(u)^*(U).$$

The Peirce decomposition

$$Z = Z_2(u) + Z_1(u) + Z_0(u), U = U_2(u) + U_1(u) + U_0(u)$$

holds. Tripotents u and v are said to be *orthogonal* if $u \in U_0(v)$ (which implies $v \in U_0(u)$) or, equivalently, $u \pm v \in \mathcal{GT}$ (see [1, Lemma 2.5]). More generally, elements a and b of U are said to be *orthogonal* if one of them belongs to $U_2(u)$ and the other belongs to $U_0(u)$ for some geometric tripotent u .

A contractive projection Q on Z is said to be *neutral* if $\|Qx\| = \|x\|$ implies $Qx = x$ for each $x \in Z$. A space Z is said to be *neutral* if, for each symmetric face F_u , the projection $P_2(u)$, corresponding to the symmetry S_u , is neutral. A space Z is said to be *atomic* if each symmetric face of Z_1 contains an extreme point.

Definition 2.3. ([1]). A *WFS space* Z is said to be *strongly facially symmetric (SFS)* if, for each norm exposed face F_u of Z_1 and each $g \in Z^*$ satisfying the conditions $\|g\| = 1$ and $F_u \subset F_g$, we have $S_u^*g = g$, where S_u is the symmetry corresponding to F_u .

Instructive examples of neutral strongly facially symmetric spaces are Hilbert spaces, the preduals of von Neumann algebras or JBW^* -algebras, and, more generally, the preduals of JBW^* -triples. Moreover, geometric tripotents correspond to nonzero partial isometries of von Neumann algebras and tripotents in JBW^* -triples (see [2]).

In a neutral strongly facially symmetric space Z , each nonzero element admits a polar decomposition [6, Theorem 4.3], that is, for $0 \neq x \in Z$ there exists a unique geometric tripotent $v = v_x$ for which $v(x) = \|x\|$ and $\langle v, x^\diamond \rangle = 0$. If $x, y \in Z$, then $x \diamond y$ if and only if $v_x \diamond v_y$ (see [1, Corollary 1.3(b) and Lemma 2.1]). A geometric tripotent $u \in U$ is said to be *minimal* if $\dim U_2(v) = 1$. Note that if u is a minimal geometric tripotent, then there exists a $z \in Z$ such that $P_2(u)(x) = u(x)z$ for all $x \in Z$.

A neutral *SFS-space* Z is said to be *point exposed (PE)* if each extreme point of the ball Z_1 is a norm exposed point [3].

The set of geometric tripotents is ordered as follows: given $u, v \in \mathcal{GT}$, we set $u \leq v$ if $F_u \subset F_v$. Note that this is equivalent to the relation $P_2(u)^*v = u$ or to the condition that $v - u$ either vanishes or is a geometric tripotent orthogonal to u (see [6, Lemma 4.2]).

3. MAIN RESULT

In what follows, Z is always assumed to be a neutral strongly facially symmetric space for which there exists a geometric tripotent e satisfying the condition $P_2(e) = I$, i.e., $Z_2(e) = Z$. We set

$$\nabla = \{u \in \mathcal{GT} : u \leq e\} \cup \{0\}.$$

As is known [6, Proposition 4.5], the set ∇ is a complete orthomodular lattice with orthocomplement $u^\perp = e - u$ with respect to the order " \leq ".

First, we prove several auxiliary lemmas.

Lemma 3.1. *If ∇ is a Boolean algebra, then, for any $u \in \nabla$, $u \neq 0$,*

- (a) $P_1(u) = 0$;
- (b) $P_2(u) = P_0(u^\perp)$.

Proof. (a) Take $u \in \nabla$, $u \neq 0$. First, let us show that

$$F_e \subseteq \overline{sp}F_u \oplus \overline{sp}F_{u^\perp}.$$

Suppose that, on the contrary, there exists an element

$$a \in F_e \setminus \overline{sp}F_u \oplus \overline{sp}F_{u^\perp}.$$

On the subspace $sp\{a\} \oplus \overline{sp}F_u \oplus \overline{sp}F_{u^\perp}$ we define a functional g by

$$g(\lambda a + x) = \lambda, \quad \lambda \in \mathbb{C}, \quad x \in \overline{sp}F_u \oplus \overline{sp}F_{u^\perp}.$$

By the Hahn — Banach theorem, g admits an extension to Z , with the same norm; we denote this extension by the same symbol g . We have

$$\|g\| = 1, \quad g(a) = 1,$$

$$(3.1) \quad g|_{\overline{sp}F_u \oplus \overline{sp}F_{u^\perp}} = 0.$$

Take a tripotent $v \in Z^*$ for which $F_g = F_v$. By virtue of (3.1), we have

$$v(a) = 1,$$

$$(3.2) \quad v|_{\overline{sp}F_u \oplus \overline{sp}F_{u^\perp}} = 0.$$

Let us show that $u \wedge v = 0$ and $u^\perp \wedge v = 0$. Indeed, if $u \wedge v \neq 0$, then there exists an $x \in Z_1$ for which $u(x) = v(x) = 1$, so that $x \in F_u$ and (3.2) implies that $v(x) = 0$, which is false. Thus,

$$u \wedge v = 0, u^\perp \wedge v = 0.$$

It follows from $a \in F_e \cap F_v$ that $e \wedge v \neq 0$. On the other hand, we have

$$e \wedge v = (u \vee u^\perp) \wedge v = (u \wedge v) \vee (u^\perp \wedge v) = 0 \vee 0 = 0,$$

because ∇ is a Boolean algebra. This contradiction implies

$$F_e \subseteq \overline{sp}F_u \oplus \overline{sp}F_{u^\perp}.$$

Since $\overline{sp}F_e = Z$, it follows that

$$(3.3) \quad Z = \overline{sp}F_u \oplus \overline{sp}F_{u^\perp}.$$

Now, let us show that $P_1(u) = 0$. Relation (3.3) means that

$$I = P_2(u) + P_2(u^\perp).$$

Hence

$$P_1(u) + P_0(u) = I - P_2(u) = P_2(u^\perp),$$

i.e.,

$$(3.4) \quad P_1(u) + P_0(u) = P_2(u^\perp).$$

The relations $P_1(u)P_0(u) = 0$ and $P_2(u^\perp) \subseteq P_0(u)$, imply $P_1(u)P_2(u^\perp) = 0$. Multiplying both sides of (3.4) by $P_1(u)$ we obtain

$$P_1(u)[P_1(u) + P_0(u)] = P_1P_2(u^\perp),$$

i.e.,

$$P_1(u) = 0.$$

(b) Note that if $u, v \in \mathcal{GT}$ are orthogonal, then, by virtue of [6, Lemma 1.8] we have

$$P_0(u + v) = P_0(u)P_0(v).$$

Since $P_1(u) = P_1(u^\perp) = \{0\}$, it follows that

$$(3.5) \quad I = P_2(u^\perp) + P_0(u^\perp)$$

and

$$\begin{aligned} I &= I \cdot I = (P_2(u) + P_0(u))(P_2(u^\perp) + P_0(u^\perp)) = \\ &= P_2(u) + P_2(u^\perp), \end{aligned}$$

i.e.,

$$(3.6) \quad I = P_2(u) + P_2(u^\perp).$$

Applying (3.5) and (3.6) we obtain $P_2(u) = P_0(u^\perp)$. This completes the proof of the lemma.

Lemma 3.2. *If ∇ is a Boolean algebra and $v_i \in \nabla$, $v_i \diamond v_j$ for $i \neq j$, $i, j = \overline{1, n}$, then*

$$P_2\left(\sum_{i=1}^n v_i\right) = \sum_{i=1}^n P_2(v_i).$$

Proof. Since $v_i \diamond v_j$, for $i \neq j$, $i, j = \overline{1, n}$, it follows by [6, Corollary 3.4] that for $i \neq j$ we have

$$P_2(v_i)P_2(v_j) = 0,$$

$$P_2(v_i)P_0(v_j) = P_2(v_i),$$

$$P_0(v_i)P_0(v_j) = P_0(v_i + v_j).$$

By virtue of Lemma 3.1 (a) we have $P_2(v_i) + P_0(v_i) = I$. Using the relations given above, we obtain

$$\begin{aligned} I = I^n &= \prod_{i=1}^n [P_2(v_i) + P_0(v_i)] = \\ &= \sum_{i=1}^n P_2(v_i) + \prod_{i=1}^n P_0(v_i) = \sum_{i=1}^n P_2(v_i) + P_0\left(\sum_{i=1}^n v_i\right), \end{aligned}$$

i.e.,

$$\sum_{i=1}^n P_2(v_i) + P_0\left(\sum_{i=1}^n v_i\right) = I.$$

The relation

$$P_2\left(\sum_{i=1}^n v_i\right) + P_0\left(\sum_{i=1}^n v_i\right) = I,$$

implies

$$P_2\left(\sum_{i=1}^n v_i\right) = \sum_{i=1}^n P_2(v_i).$$

This proves the lemma.

In what follows, we assume that Z is a PE atomic neutral strongly facially symmetric space and there exists a geometric tripotent e for which $Z_2(e) = Z$.

Let T be a maximal family of mutually orthogonal minimal geometric tripotents from Z^* , i.e., $T = \{v_i : v_i \diamond v_j, i \neq j, i, j \in J\}$, where each v_i is minimal and no minimal geometric tripotent is orthogonal to all $v_i, i \in J$. Such a family exists by Zorn's Lemma.

Lemma 3.3. *If T is a maximal family of mutually orthogonal minimal geometric tripotents from Z^* and $v = \sup\{v_i : v_i \in J\}$, then $v = e$.*

Proof. Suppose that $v \neq e$. Then $v^\perp \neq 0$. Since Z is a PE atomic SFS -space, it follows that there exists a norm exposed point x in F_{v^\perp} . The geometric tripotent u corresponding to x is minimal. Therefore, $u \leq v^\perp$. The relation $v^\perp \leq v_i^\perp$ implies $u \leq v_i^\perp$ for $i \in J$. Hence $u \diamond v_i, i \in J$. This contradicts the maximality of the family T . Thus $v = e$, which completes the proof of the lemma.

Lemma 3.4. *If T is a maximal family of mutually orthogonal minimal geometric tripotents from Z^* , then T separates the points of Z , i.e., for any $x \in Z, x \neq 0$ there exists a $v \in T$ for which $v(x) \neq 0$.*

Proof. Let T be a maximal family of mutually orthogonal minimal geometric tripotents from Z^* . According to Lemma 3.3 we have $\sup\{v_i : v_i \in J\} = e$.

Suppose that there exists a point $x \in Z$ with $\|x\| = 1$ such that $v_i(x) = 0$ for all $i \in J$. The minimality of each v_i implies the existence of a $z_i \in F_{v_i}$ such that $P_2(v_i)(x) = v_i(x)z_i$ for any $x \in Z$. Since $v_i(x) = 0$, it follows that $P_2(v_i)(x) = 0$, and since $P_1(u) = 0$, it follows that $x \in Z_0(v_i)$ for all $i \in J$. By virtue of Lemma 3.1 (a) we have $P_2(v_i^\perp) = P_0(v_i)$. Therefore, $x \in Z_2(v_i^\perp)$ for all $i \in J$. Take a minimal geometric tripotent v for which $x \in F_v$. We have $v \diamond v_i$ for all $i \in J$. Thus, $v \diamond \sup v_i$, i.e., $v \diamond e$, whence $x \in P_0(e)$. It follows from $P_0(e) = 0$ that $x = 0$. This contradicts the assumption $x \neq 0$. Thus, T separates the points of Z , as required.

Let $T = \{v_i\}_{i \in J}$ be a maximal family of mutually orthogonal minimal geometric tripotents from Z^* , and let

$$\ell_1(T) = \left\{ \{\lambda_i\}_{i \in J} : \sum_{i \in J} |\lambda_i| < +\infty \right\}.$$

Then $\ell_1(T)$ is a Banach space with respect to the norm

$$\|\{\lambda_i\}_{i \in J}\| = \sum_{i \in J} |\lambda_i|.$$

The following theorem is the main result of this paper; it describes strongly facially symmetric spaces isometrically isomorphic to preduals of atomic commutative von Neumann algebras.

Theorem 3.5. *Suppose that Z is a PE neutral atomic strongly facially symmetric space and there exists a geometric tripotent e for which $Z_2(e) = Z$. If ∇ is a Boolean algebra, then Z is isometrically isomorphic to the space $\ell_1(T)$, where T is a maximal family of mutually orthogonal minimal geometric tripotents from Z^* .*

Proof. Let $z_i \in F_e$ be such that $v_i(z_i) = 1$. Then, for any $x \in Z$ we have $P_2(v_i)(x) = v_i(x)z_i$, where $v_i(x) \in \mathbb{R}, i \in J$. Let us show that

$$x = \sum_{i \in J} v_i(x)z_i$$

and

$$\|x\| = \sum_{i \in J} |v_i(x)|.$$

For $i_1, \dots, i_n \in J$, we have

$$\begin{aligned} \sum_{k=1}^n |v_{i_k}(x)| &= \sum_{k=1}^n |v_{i_k}(x)| \|z_{i_k}\| = \\ &= [z_{i_k} \diamond z_{i_s}, k \neq s] = \\ &= \left\| \sum_{k=1}^n v_{i_k}(x)z_{i_k} \right\| = \left\| \sum_{k=1}^n P_2(v_{i_k})(x) \right\| = \end{aligned}$$

$$= \left\| P_2 \left(\sum_{k=1}^n v_{i_k} \right) (x) \right\| \leq \|x\|,$$

i.e.,

$$\sum_{k=1}^n |v_{i_k}(x)| \leq \|x\|.$$

Therefore, the series $\sum_{i \in J} |v_i(x)|$ converges, and $\sum_{i \in J} |v_i(x)| \leq \|x\|$. Next, for $j \in J$ we have

$$v_j \left(x - \sum v_i(x) z_i \right) = v_j(x) - v_j(x) = 0.$$

Since T separates the points of Z , it follows that $x = \sum_{i \in J} v_i(x) z_i$. Thus, the correspondence

$$x \in Z \mapsto \{v_i(x)\}$$

is an isometric isomorphism between Z and $\ell_1(T)$. This completes the proof of the theorem.

Corollary 3.6. *Suppose that Z is a real finite-dimensional neutral strongly facially symmetric space and there exists a geometric tripotent e for which $Z_2(e) = Z$. If ∇ is a Boolean algebra, then Z is isometrically isomorphic to the space \mathbb{R}^n with norm*

$$\|z\| = |z_1| + \dots + |z_n|.$$

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